

# Efficient Calculation of Directivity Indices for Certain Three-Dimensional Arrays

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## PREFACE

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# EFFICIENT CALCULATION OF DIRECTIVITY INDICES FOR CERTAIN THREE-DIMENSIONAL ARRAYS

## INTRODUCTION

Directivity Index (DI) may be defined as a measure of the improvement in the signal-to-noise power ratio (S/N) that a beamformed array provides in an *ideal isotropic noise field* with a perfectly correlated signal, relative to an omni-directional array element in the free-field, i.e.,

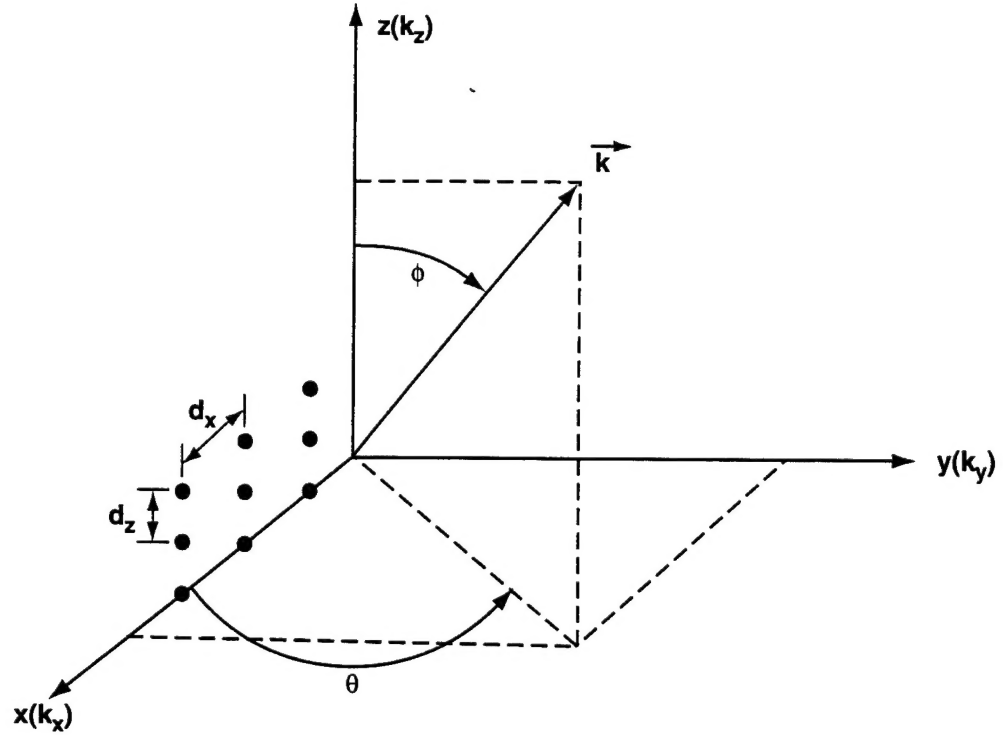
$$DI = 10 \log DF = 10 \log \frac{(S/N)_{array\ output}}{(S/N)_{free-field\ omni-element}}. \quad (1)$$

For an array with arbitrary geometry and element locations, the beamformed array single-frequency amplitude response<sup>1</sup> may be written as

$$A(\theta, \phi) = \sum_{t=1}^T w_t g_t(\theta, \phi) \exp(i(\vec{k} - \vec{k}_s) \cdot \vec{r}_t), \quad (2)$$

where  $A(\theta, \phi)$  is the array's angular response to an arrival from azimuthal angle  $\theta$  and polar angle  $\phi$ ,  $T$  is the total number of array elements,  $w_t$  and  $g_t$  are, respectively, the amplitude shading and element angular sensitivity of the  $t$ -th array element,  $\vec{k}$  is the wavevector corresponding to an arriving acoustic planewave, and  $\vec{k}_s$  is the wavevector to which the array is steered. The corresponding steering angles are  $\theta_s, \phi_s$ . The  $t$ -th array element position is defined by the coordinate vector  $\vec{r}_t$ .

A spatial coordinate system (with corresponding wavevectors) for an array oriented in the  $xz$ -plane is illustrated in figure 1, along with the definition of spherical angles  $\theta, \phi$ . The programs supplied in the appendices of this report, to calculate directivity, use the geometry of figure 1 to define input parameters. For the programs **planar-xz-equal.f** and **planar-xz-grid.f**, a rectangular array in the  $xz$ -plane is assumed. Program **planar-xz-grid.f** allows for arbitrary element grid spacing, that is,  $d_x$  or  $d_z$  may vary from element-to-element. The program **volumetric-grid.f** assumes an array configuration which allows for cylindrical curvature about the  $x$ -axis. At each fixed array row  $x_\ell$ , elements are located at the same positions  $\{y_n, z_n\}$ ; if all  $y_n$  are equal, the array becomes a plane parallel to the  $xz$ -plane.



**Figure 1. Array Coordinate System and Spherical Angles Denoting Incident Planewave Arrival Direction  $\theta, \phi$**

With the aid of equations (1) and (2), the components which comprise the calculation of the directivity index can be written as

$$S_{ff-omni} = S, \quad (3a)$$

where  $S$  is defined as the given signal power level, while

$$S_{array} = S |A(\theta_s, \phi_s)|^2, \quad (3b)$$

$$N_{ff-omni} = \int_0^{2\pi} \int_0^\pi N \sin(\phi) d\phi d\theta, \quad (3c)$$

where  $N$  is defined as the isotropic noise level, while

$$N_{array} = \int_0^{2\pi} \int_0^\pi N |A(\theta, \phi)|^2 \sin(\phi) d\phi d\theta. \quad (3d)$$

Noting that equation (3c), is simply  $4\pi N$ , and canceling the signal and isotropic noise levels, equation (1) reduces to

$$DI = 10 \text{ Log} \left\{ \frac{4\pi |A(\theta_s, \phi_s)|^2}{\int_0^{2\pi} \int_0^\pi |A(\theta, \phi)|^2 \sin(\phi) d\phi d\theta} \right\}, \quad (4)$$

where  $|A(\theta_s, \phi_s)|^2$  is the array power response at steering angles  $\theta_s, \phi_s$ .

Equation (3d) indicates an integral with limits that account for all acoustic arrival angles. Hence, the directivity expression above assumes an unbaffled, or free-field, array configuration. In a baffle configuration,  $N_{array}$  would be reduced, resulting in an increased directivity. However, the programs supplied here include directional (such as cosine to a power) element sensitivities,  $g_t(\theta, \phi)$  in (2), which inherently provide the gains realized from a baffle. There is no need to correct the directivity values yielded by the programs to account for baffling of isotropic free-field noise.

For a line array with constant inter-element spacing, the double integral in equation (4) may be evaluated exactly at the frequency for which spacing  $d_x = \lambda/2$ , regardless of the steering direction. This leads to

$$DI = 10 \log \left\{ \frac{\left( \sum_{t=1}^T w_t \right)^2}{\sum_{t=1}^T w_t^2} \right\}. \quad (5)$$

Accurate directivity predictions for arbitrary array configurations, at general frequencies of interest, require numerically integrating the double integral given in equation (4) and calculating the maximum array power response,  $|A(\theta_m, \phi_m)|^2$ . It should be noted that the maximum array response may not occur precisely at the steering angles  $(\theta_s, \phi_s)$ . Array



curvature, unequal array element spacing, element sensitivity, and shading weights act to shift the maximum response axis from  $(\theta_s, \phi_s)$  to  $(\theta_m, \phi_m)$ . The programs supplied here have global and local search procedures which obtain the peak power response and the corresponding angles  $(\theta_m, \phi_m)$ .

The remainder of this report will discuss accurate and efficient numerical integration methods for evaluating the double integral of equation (4). The focus is to reduce the number of evaluations required of the integrand, in order to achieve rapid convergence and accuracy with a minimum of computational effort.

For very small partitions, the Riemann sum will provide a reasonable approximation of the definite integrals in equation (4). Indeed, if an exceedingly large number of narrow rectangles (or augmented partitions) are used in the Riemann sum, the approximation would be quite accurate. This implies, however, an enormous amount of function evaluations, each of which is computationally time-consuming. For a large number of array elements, on the order of thousands, such a simplistic approach would be unfeasible. Vector and matrix operations used in high-performance computation software, such as MATLAB<sup>®</sup>, do provide quick function evaluations and have built-in numerical integration routines. However, these computational benefits can be reduced by the amount of RAM available on a given computer system. Swapping memory space between a system's hard drive and RAM can be inefficient; resulting in lengthy integration times. The goal then is to use an integration method which yields a highly accurate integral approximation with a minimum number of function evaluations.

Due to the periodicity and analyticity of the integrand  $|A(\theta, \phi)|^2$  in azimuthal angle  $\theta$ , an elementary closed-type Trapezoidal formula was used for evaluating the outer  $\theta$  integral in equation (4) over a full period  $(0, 2\pi)$ . The inner integral, over polar angle  $\phi$ , uses a Gauss-Legendre quadrature formula of order  $m$ , where  $m$  can vary over the twelve values (16, 24, 32, 48, 64, 96, 128, 192, 256, 384, 512, 768). The Gauss-Legendre method is generally significantly more accurate than an equal-interval formula, such as the Trapezoidal rule, for a given number of function evaluations. However, Gauss quadrature was deliberately not chosen for the outer  $\theta$ -integral due to the particular integration limits and the integrand periodicity and analyticity in  $\theta$ . The Trapezoidal rule is extremely efficient for integration of an analytic periodic integrand, when conducted over a full period.<sup>2</sup> Generally, efficient numerical evaluation of multi-dimensional integrals requires examination of the nature of the integrand and the integration limits; rarely is one procedure optimum for all integrals.

## FACTORIZATION OF ARRAY RESPONSE

The array amplitude response  $A(\theta, \phi)$  was given in equation (2) for the general three-dimensional array with arbitrary element locations, weights, and element responses, where  $T$  is the total number of elements in the array. In the general case, the sum of  $T$  terms can be evaluated only by calculating every individual complex term and summing up those  $T$  quantities. This is a particularly time-consuming task, especially when  $T$  is of the order of 10,000 or more. When coupled with the fact that the directivity index requires computation of a double integral, namely equation (4), which must be repeated for each different frequency and/or steering angle, the computational burden becomes excessive. Furthermore, as the number  $T$  of elements increases, the array power response  $|A(\theta, \phi)|^2$  becomes even sharper in angles  $\theta, \phi$ , thereby requiring still finer evaluation of the integrand in equation (4), in order to retain accuracy in the final DI calculation.

In this section, we will significantly reduce this computational burden for a particular class of arrays and weights, by factoring the amplitude response  $A(\theta, \phi)$  into a product of two sums which have far fewer total terms than  $T$ . For example, an array of 50 by 200 elements will require evaluation of  $50 + 200 = 250$  terms instead of  $50 * 200 = 10,000$  terms, a savings in execution time by a factor of 40. Additional savings in time are achieved by precomputing and storing quantities that are used repeatedly in the program for calculation of the DI. This storage requirement is minimal, even for the very large arrays of interest here.

## ELEMENT LOCATIONS

We use the coordinate system illustrated in figure 1. Receiving elements will be located only on  $L$  planes of fixed  $x$ , namely, at  $\{x_\ell\}$  for  $1 \leq \ell \leq L$ , where each  $x_\ell$  is arbitrary. Thus, these planes need not be equally spaced in  $x$ .

At each  $x$ -coordinate  $x_\ell$ ,  $N$  elements are then located at the points  $y_n, z_n$  for  $1 \leq n \leq N$ ; these pairs  $\{y_n, z_n\}$  are identical for all  $x_\ell$ . However, each location  $y_n, z_n$  is arbitrary. This generality allows for the array to take on a cylindrical shape, for example; other more general three-dimensional arrays are also allowed. In the special case where the  $\{y_n\}$  are all equal, this becomes a planar array, parallel to the  $xz$ -plane; however, the  $L$  locations  $\{x_\ell\}$  and the  $N$  locations  $\{z_n\}$  are still arbitrary.

The array is seen to contain a total of  $T = L N$  elements, in  $L$  parallel planes of  $N$  elements each, with an identical repeated (arbitrary) configuration in each  $x$ -plane. This is the general configuration considered here. Any three-dimensional array which cannot be put into this form will not be covered by the following analysis and simplifications.

Each element at  $x_\ell, y_n, z_n$  has its normal in the  $yz$ -plane, at an angle  $\psi_n$  relative to the  $z$ -axis; see figure 2. That is, the element lies in the  $yz$ -plane for every element in the array; however, the element orientation angle  $\psi_n$  can vary with location  $y_n, z_n$ , but not with location  $x_\ell$ . Thus, the direction vector of the  $n$ -th element boresight is  $0 \mathbf{i} + \sin(\psi_n) \mathbf{j} + \cos(\psi_n) \mathbf{k}$  for the element at  $x_\ell, y_n, z_n$ .

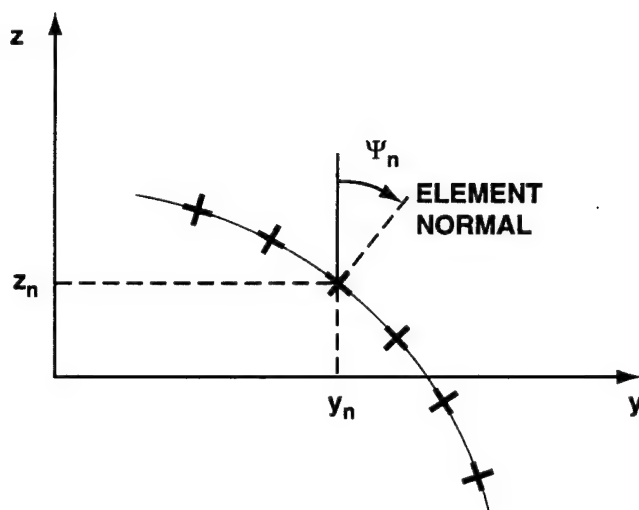


Figure 2. Element Locations at a Fixed  $x_\ell$

## ELEMENT RESPONSES

The cosine of the angle  $B_n(\theta, \phi)$  between the  $n$ -th element normal and a general arrival direction  $\theta, \phi$  is given by the dot product

$$\begin{aligned} \cos B_n(\theta, \phi) &= [0 \mathbf{i} + \sin(\psi_n) \mathbf{j} + \cos(\psi_n) \mathbf{k}] \cdot \\ &\cdot [\cos(\theta) \sin(\phi) \mathbf{i} + \sin(\theta) \sin(\phi) \mathbf{j} + \cos(\phi) \mathbf{k}] = \\ &= \sin(\psi_n) \sin(\theta) \sin(\phi) + \cos(\psi_n) \cos(\phi). \end{aligned} \quad (6)$$

Notice that this quantity is independent of  $\ell$  and  $x_\ell$ , due to the element locations and boresight orientations.

We are interested here in relative element responses of the form

$$r_n(\theta, \phi) = [\cos B_n(\theta, \phi)]^\nu \quad \text{for } |B_n(\theta, \phi)| \leq \pi/2, \quad (7)$$

where  $\nu$  is a power under our control. By means of the procedure presented in Appendix A, the element responses can be approximated by

$$r_n(\theta, \phi) \equiv \exp \left[ -\nu \left( u_n + \frac{1}{2} u_n^2 + \frac{1}{3} u_n^3 + \cdots + \frac{1}{I} u_n^I \right) \right], \quad (8)$$

where  $I$  is an integer under our control, and

$$u_n \equiv 1 - \cos B_n(\theta, \phi) = 1 - \sin(\psi_n) \sin(\theta) \sin(\phi) - \cos(\psi_n) \cos(\phi). \quad (9)$$

Here, we also used equation (6). We will use the element response  $r_n(\theta, \phi)$  to an arrival from direction  $\theta, \phi$ , as given by equation (8) and (9), for all the calculations herein. The region of applicability is  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \phi \leq \pi$ .

## ELEMENT WEIGHTS

A multiplicative weight  $\alpha_n \beta_\ell$  is applied at general element location  $x_\ell, y_n, z_n$  for  $1 \leq \ell \leq L, 1 \leq n \leq N$ . The total effect of weighting and element response is therefore given by

$$w_{n\ell}(\theta, \phi) = \alpha_n \beta_\ell r_n(\theta, \phi). \quad (10)$$

It is important to observe that this total effective element response is factorable in the variables  $n$  and  $\ell$ ; it need not be factorable in  $\theta$  and  $\phi$ .

## ARRAY AMPLITUDE RESPONSE

For notational simplicity, the following definitions will be used below:

$$\underline{x}_\ell \equiv 2\pi x_\ell / \lambda, \quad \underline{y}_n \equiv 2\pi y_n / \lambda, \quad \underline{z}_n \equiv 2\pi z_n / \lambda, \quad \exp i(x) \equiv \exp(ix), \quad (11)$$

where  $\lambda$  is the wavelength of the single-frequency arriving planewave. When the features above are incorporated into the array amplitude response (2), it can be rewritten in the form

$$A(\theta, \phi) = \sum_{n=1}^N \sum_{\ell=1}^L w_{n\ell}(\theta, \phi) \exp i \left[ \underline{x}_\ell \cos \theta \sin \phi + \underline{y}_n \sin \theta \sin \phi + \underline{z}_n \cos \phi \right. \\ \left. - \left( \underline{x}_\ell \cos \theta_s \sin \phi_s + \underline{y}_n \sin \theta_s \sin \phi_s + \underline{z}_n \cos \phi_s \right) \right], \quad (12)$$

where  $\theta_s, \phi_s$  is the steering direction, and  $\theta, \phi$  is the planewave arrival direction. Finally, when the particular form of weighting in (10) is employed, amplitude response (12) further simplifies to the desired factored form

$$A(\theta, \phi) = \sum_{n=1}^N \alpha_n r_n(\theta, \phi) \exp i \left( \underline{y}_n \sin \theta \sin \phi + \underline{z}_n \cos \phi - a_n \right) \times \\ \times \sum_{\ell=1}^L \beta_\ell \exp i \left( \underline{x}_\ell \cos \theta \sin \phi - b_\ell \right), \quad (13)$$

where parameters

$$a_n \equiv y_n \sin \theta_s \sin \phi_s + z_n \cos \phi_s, \quad b_\ell \equiv x_\ell \cos \theta_s \sin \phi_s. \quad (14)$$

The computation of amplitude response  $A(\theta, \phi)$  by means of (13) requires an  $N$ -fold summation on  $n$  and a  $L$ -fold summation on  $\ell$ , which constitutes a total of  $N + L$  complex terms. When these complex operations are completed, the magnitude-squared operation yielding power response  $|A(\theta, \phi)|^2$  is taken. The complex function  $\exp i(x) \equiv \exp(ix) = \cos(x) + i \sin(x)$ .

## NUMERICAL EVALUATION OF DOUBLE INTEGRAL

The major computation required in DI calculation (4) is the double integral

$$V \equiv \int_0^\pi d\phi \sin \phi \int_0^{2\pi} d\theta |A(\theta, \phi)|^2. \quad (15)$$

When the factorized expression (13) is substituted in (15), the result is expressible in the form

$$V = \int_0^\pi d\phi \sin \phi \int_0^{2\pi} d\theta \left| \sum_{\ell=1}^L \beta_\ell \exp i(x_\ell \cos \theta \sin \phi - b_\ell) \right|^2 \times \quad (16)$$

$$\times \left| \sum_{n=1}^N \alpha_n r_n(\theta, \phi) \exp i(y_n \sin \theta \sin \phi + z_n \cos \phi - a_n) \right|^2.$$

Element response  $r_n(\theta, \phi)$  is given by (8) and (9).

The integrand in (16) is analytic in  $\theta$  and  $\phi$ . Furthermore, it has period  $2\pi$  in  $\theta$ , and the  $\theta$  integral is over this complete interval. This makes the  $\theta$  integral in (16) a good candidate for numerical evaluation via the trapezoidal rule [2; pp. 53-59]. On the other hand, the  $\phi$  integral is only over  $(0, \pi)$ , which is half the period in  $\phi$ . This suggests the Gauss-Legendre rule for numerical integration on  $\phi$ . This mixed procedure has been utilized for the approximate evaluation of (16).

The specific formula for  $M$ -point Gauss-Legendre integration of function  $f(x)$  over interval  $(a, b)$  has the general form

$$\begin{aligned}
\int_a^b dx f(x) &= \frac{b-a}{2} \int_{-1}^1 dt f\left(\frac{b+a}{2} + \frac{b-a}{2} t\right) \equiv \frac{b-a}{2} \sum_{m=1}^M w_m f\left(\frac{b+a}{2} + \frac{b-a}{2} t_m\right) = \\
&= \frac{b-a}{2} \sum_{m=1}^M w_m f(x_m); \quad x_m \equiv \frac{b+a}{2} + \frac{b-a}{2} t_m.
\end{aligned}
\tag{17}$$

The quantities  $\{t_m\}$  are the  $M$  zero locations of the  $M$ -th order Legendre polynomial  $P_M(x)$ , and are symmetrically located in the  $(-1,1)$  interval. The  $M$  Gauss weights  $\{w_m\}$  are all positive. These quantities ( $t_m$  and  $w_m$  for  $M$  up to 768) are required as input to the numerical programs provided in the appendices. Gauss-Legendre weights and zero-locations are available in many references.<sup>3</sup> When combined with the trapezoidal rule for integration on  $\theta$ , there follows for (16), with  $\phi_m \equiv \frac{1}{2}\pi + \frac{1}{2}\pi t_m$ ,  $\theta_k = 2\pi k/K$ ,

$$\begin{aligned}
V &\equiv \frac{\pi}{2} \sum_{m=1}^M w_m \sin \phi_m \frac{2\pi}{K} \sum_{k=1}^K \left| \sum_{\ell=1}^L \beta_{\ell} \exp i(\underline{x}_{\ell} \cos \theta_k \sin \phi_m - b_{\ell}) \right|^2 \times \\
&\times \left| \sum_{n=1}^N \alpha_n r_n(\theta_k, \phi_m) \exp i(\underline{y}_n \sin \theta_k \sin \phi_m + \underline{z}_n \cos \phi_m - a_n) \right|^2,
\end{aligned}
\tag{18}$$

where the element response is now expressible as

$$r_n(\theta_k, \phi_m) = \exp \left[ -v \left( u + \frac{1}{2} u^2 + \frac{1}{3} u^3 + \dots + \frac{1}{I} u^I \right) \right]
\tag{19}$$

with

$$u \equiv 1 - \sin \psi_n \sin \theta_k \sin \phi_m - \cos \psi_n \cos \phi_m.
\tag{20}$$

For trapezoidal integration, the interval  $(0, 2\pi)$  was cut into  $K$  panels, each of width  $2\pi/K$ .

However, the two usual edge weights of  $1/2$  have been combined into a single unity weight at  $\theta = 2\pi$ , by making use of the periodicity in  $\theta$ . Hence, the summation on  $k$  in (18) goes from 1 to  $K$ , using unity weights throughout.

Expression (18) is the final result to be used for numerical evaluation of the double integral for  $V$  in (15), when the array is three-dimensional. Quantities such as  $\sin\theta_k$ ,  $\sin\phi_m$ , etc., that can be done once and for all, have been precomputed and stored for access in (18) as required, in order to minimize the execution time. A program for the evaluation of (18) is available in Appendix B under the title **volumetric-grid.f**.

For a nonisotropic noise field which depends only on polar angle  $\phi$ , say with power dependence  $D(\phi)$ , the only changes required are to insert  $D(\phi)$  after  $\sin\phi$  in (15), and to insert  $D(\phi_m)$  after  $\sin\phi_m$  in (18). Of course, the integral of product  $\sin(\phi) D(\phi)$  over  $(0, \pi)$  must also be evaluated.

### SPECIALIZATION TO PLANAR ARRAY, ARBITRARY GRID

When all the  $y$ -coordinates  $\{y_n\}$  are zero (or constant), the array reduces to a planar array parallel to the  $xz$ -plane; however, the element locations can still be irregular, namely at each intersection  $x_\ell, z_n$  for  $1 \leq \ell \leq L$ ,  $1 \leq n \leq N$ . Every  $x$ -coordinate  $x_\ell$  is arbitrary, and every  $z$ -coordinate  $z_n$  is arbitrary in this grid structure.

When the boresight of every individual element is perpendicular to the array plane, angle  $\psi_n$  in figure 2 is equal to  $\pi/2$  for all  $n$ . This makes  $u = 1 - \sin\theta_k \sin\phi_m$  in (20), which is independent of  $n$ ; therefore,  $r_n(\theta_k, \phi_m)$  in (19) is also independent of  $n$ , allowing it to be factored out of the  $n$  summation in (18).

Also, when all the  $\{y_n\}$  are zero, the latter  $\exp i$  term in (18) becomes independent of  $k$ , allowing the entire magnitude-squared quantity in the second line of (18) to be factored out of the  $k$  summation. The end result is the simplified form

$$V \equiv \frac{\pi}{2} \sum_{m=1}^M w_m \sin\phi_m \left| \sum_{n=1}^N \alpha_n \exp i(z_n \cos\phi_m - a_n) \right|^2 \times \quad (21)$$

$$\times \frac{2\pi}{K} \sum_{k=1}^K \left| r(\theta_k, \phi_m) \right|^2 \left| \sum_{\ell=1}^L \beta_\ell \exp i(x_\ell \cos\theta_k \sin\phi_m - b_\ell) \right|^2,$$

where now



$$\left| r(\theta_k, \phi_m) \right|^2 = \exp \left[ -2 \left( u + \frac{1}{2} u^2 + \frac{1}{3} u^3 + \dots + \frac{1}{I} u^I \right) \right], \quad (22)$$

with

$$u = 1 - \sin \theta_k \sin \phi_m. \quad (23)$$

It has been found that these manipulations and simplifications result in approximately 30 percent savings in computation time relative to the more general volumetric form (18). A program for simplified form (21) is listed in Appendix C under the title planar-xz-grid.f, where the xz qualifier emphasizes the fact that the planar array is parallel to the xz-plane.

### SPECIALIZATION TO PLANAR ARRAY, EQUAL SPACINGS

A useful additional simplification is possible when the planar array has equally spaced elements in both the x- and z-coordinates. Specifically, suppose that the element locations are given by

$$x_\ell = \ell d_x \quad \text{for } 1 \leq \ell \leq L, \quad z_n = n d_z \quad \text{for } 1 \leq n \leq N. \quad (24)$$

Substitution of these values into (21), and reference to (11) and (14), results in the modified form

$$V \equiv \frac{\pi}{2} \sum_{m=1}^M w_m \sin \phi_m \left| \sum_{n=1}^N \alpha_n \exp i \left( \frac{2\pi}{\lambda} d_z (\cos \phi_m - \cos \phi_s) n \right) \right|^2 \frac{2\pi}{K} \times \quad (25)$$

$$\sum_{k=1}^K \left| r(\theta_k, \phi_m) \right|^2 \left| \sum_{\ell=1}^L \beta_\ell \exp i \left( \frac{2\pi}{\lambda} d_x (\cos \theta_k \sin \phi_m - \cos \theta_s \sin \phi_s) \ell \right) \right|^2$$

where (22) and (23) are still relevant.

The reduction in computation time is now obtained by taking advantage of the recursive capability of the expi function:

$$\exp i[n\gamma] = \exp[i n \gamma] = \exp[i \gamma] \exp[i(n-1)\gamma] = \exp i[\gamma] \exp i[(n-1)\gamma]. \quad (26)$$

Thus, one complex multiplication gives the next term necessary in the  $n$  or  $\ell$  summations in (25), without the need for any storage.

It has been found that these manipulations and simplifications result in approximately 60 percent additional savings in computation time relative to the more general planar form (21). A program for simplified planar form (25) is listed in Appendix D under the title planar-xz-equal.f, where the xz-qualifier again emphasizes the fact that the planar array is parallel to the xz-plane.

## RESULTS

Directivity calculations, using a low-order quadrature formula, are given in Ref. [4] for a 9-element line array, a 60-element planar array and a 21-element cylindrical array. For arrays with relatively few elements, such as these, there is little need to be concerned with optimal integration procedures. Computationally efficient integration routines can provide order-of-magnitude speed up times, but this is of little importance if the original calculation only takes a few seconds. However, the arrays considered here are large, with dense element spacings, and the frequency and steering angular dependence of the directivity must be examined in detail. Therefore, it is essential to have an efficient means to accurately evaluate the two-fold integral of equation (4).

Table 1 below summarizes the planar and conformal array lay-outs used for preliminary numerical directivity calculations. Directivity was calculated at a number of frequencies and assumed a sound speed of 4,900 ft/sec.

**Table 1. Planar and Volumetric Array Input Parameters for Directivity Calculations**

Parameter	Input Field	planar-xz equal.f	volumetric- grid.f
No. of Elements along x-axis	$L = 200$	√	√
No. of Elements along z-axis	$N = 50$	√	X
Element spacing along x-axis	$\Delta_x = 3.5$ in.	√	√
Element spacing along z-axis	$\Delta_z = 3.6$ in.	√	X
Element spacing (arc length)	$d_s = 3.6$ in.	X	√
$\theta_s$ -steering angle (azimuthal)	$\theta_s = \pi/2$ rads	√	√
$\phi_s$ -steering angle (polar)	$\phi_s = \pi/2$ rads	√	√
$\theta$ -angle increment	$\Delta\theta = 0.0001$ rads	√	√
$\phi$ -angle increment	$\Delta\phi = 0.0001$ rads	√	√

X = Input not required.

For both examples, the total number of array elements was 10,000. The angular increment parameters shown are necessary in order to obtain a reasonable estimate of the maximum array response,  $|A(\theta_m, \phi_m)|^2$ . In general, the number of angular samples is related to the array length, and frequency, i.e., we must require angular sampling of the order

$$(\Delta\theta, \Delta\phi) \leq \frac{\lambda}{4L_e}, \quad (27)$$

where  $\lambda$  is the acoustic wavelength and  $L_e = \left(L_x^2 + L_y^2\right)^{\frac{1}{2}}$ . For the numerical programs provided here, this  $\theta$  and  $\phi$  sampling requirement corresponds to the restriction  $K > 2\pi L_e/\lambda$  for  $\theta$ -sampling, where  $K$  is the program parameter  $kc$ , and  $M > \frac{\pi^2}{2} L_e/\lambda$  for  $\phi$ -sampling, where  $M$  is the program parameter  $mc$ . Additional input parameters, common to each program, are given in Appendix B.

The computed directivity is given in table 2. As expected, the planar array geometry yields slightly greater directivity than that of the conformal array; the differences are due to array curvature. With factorization, it took 154 seconds to compute the directivity value 38.57 dB (at 4,000 Hz) using the program *volumetric-grid.f* on a SUN SPARC Station 10. Without factorization, the time required to compute this value was on the order of 1 hour, 10 minutes.

*Table 2. Directivity DI (dB) Calculated From planar-xz-equal.f and volumetric-grid.f*

Frequency (Hz)	DI planar-xz-equal.f (dB)	DI volumetric-grid.f (dB)
4,000	38.71	38.57
5,000	40.63	40.49
6,000	42.21	42.06
7,000	43.54	43.39
8,000	44.69	44.54

As previously discussed, array gains in nonisotropic noise fields (having variations in polar angle only) can be evaluated using the programs supplied here. In many ocean environments, such as in the Bering or Norwegian Seas, a distinctive and well-defined notch or dipole-like directivity pattern is present in ambient noise. This variation has been described as a vertical noise profile<sup>5</sup> and may be modeled using the continuous expression

$$D(\phi) = 1 - A_d \exp\left\{-\frac{(\phi - \phi_d)^2}{2\sigma_d^2}\right\}. \quad (28)$$

Figure 3 illustrates a typical nonisotropic noise power dependence for the parameters  $A_d = 0.9$ , which controls the notch depth,  $\phi_d = \pi/2$ , the notch location, and  $\sigma_d = 0.1$ , the notch width.

Table 3 provides the array gain (AG) calculated from planar-xz-equal.f and volumetric-grid.f assuming the above vertical noise power distribution. The gain over isotropic noise varies from 6.8 dB to 7.9 dB; certainly, the vertical noise profile has a significant effect on AG calculations.

**Table 3. Array Gain (AG) for Nonisotropic Noise Calculated From the planar-xz-equal.f and volumetric-grid.f Programs**

Frequency (Hz)	DI planar-xz-equal.f (dB)	DI volumetric-grid.f (dB)
4,000	45.54	45.32
5,000	47.87	47.65
6,000	49.73	49.52
7,000	51.28	51.07
8,000	52.60	52.38

## APPROXIMATION FOR DIRECTIVITY INDEX

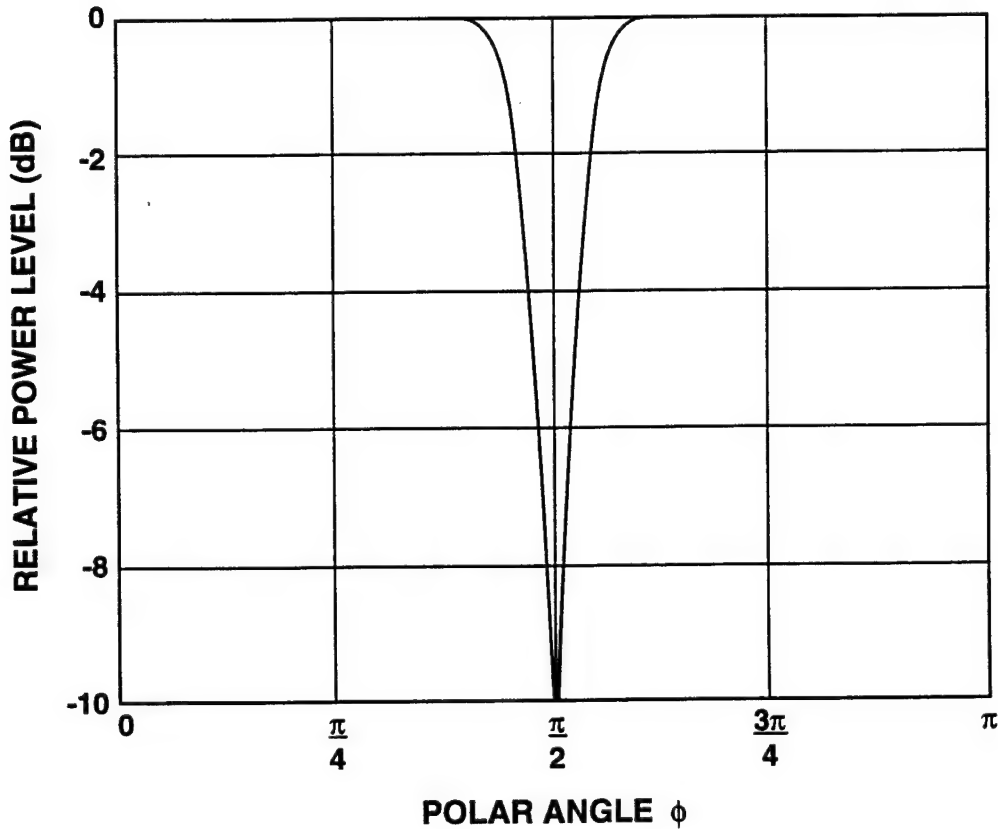
The approximation  $10 \log(4\pi A/\lambda^2)$  for the directivity index DI of an equi-spaced baffled planar array is only applicable for an unweighted array with half-omni directionality and an isotropic noise field. However, the restriction to an unweighted array can be eliminated if the area  $A$  is replaced by the product of effective lengths in the  $x$ - and  $z$ -directions. That is, we have, more generally, for a planar array with spacings  $d_x$  and  $d_z$ ,

$$DI \cong 10 \log \left( \frac{4\pi E_x E_z}{\lambda^2} \right), \quad (29)$$

where effective lengths  $E_x$  and  $E_z$  are given, respectively, by

$$E_x \equiv d_x \frac{\left( \sum_{\ell=1}^L \beta_{\ell} \right)^2}{\sum_{\ell=1}^L \beta_{\ell}^2}, \quad E_z \equiv d_z \frac{\left( \sum_{n=1}^N \alpha_n \right)^2}{\sum_{n=1}^N \alpha_n^2}. \quad (30)$$

For the isotropic noise example above, at 4,000 Hz, but now with Hanning weighting in the x-direction and flat weighting in the z-direction, the DI as found from the approximation is 33.88 dB, whereas the exact value turns out to be 33.85 dB, a discrepancy of only 0.03 dB. And for Hanning weighting in both directions, the approximate DI is 32.12 dB, whereas the exact value is 32.00 dB, a difference of 0.12 dB.



**Figure 3. Typical Variation of Vertical Noise Directionality Assuming Noise Power Dependence  $D(\phi)$  With  $A_d = 0.9$ ,  $\phi_d = \pi/2$ , and  $\sigma_d = 0.1$**

## REFERENCES

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2. P. J. Davis and P. Rabinowitz, *Numerical Integration*, Blaisdell Publishing Company, Waltham, MA, 1967.
3. M. Abramowitz and I. E. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, U.S. GPO, 1972.
4. D. Lee, "Computation of Beam Patterns and Directivity Indices for Three-Dimensional Arrays with Arbitrary Element Spacings," NUSC Technical Report 4687, Naval Underwater Systems Center, New London, CT, 22 February 1974.
5. T. Revello, "Beamformed Environment Noise," Presentation to the Naval Undersea Warfare Center Detachment, New London, CT, 11 December 1995.

## APPENDIX A — APPROXIMATION TO THE ELEMENT RESPONSE

The power response of a baffled directive receiving element can often be fairly well approximated by  $\cos(x)$  for  $|x| < \pi/2$ , where  $x$  is the angle between the element boresight angle and the planewave arrival angle. For  $|x| > \pi/2$ , the response is ideally zero. The discontinuity in slope of the response at  $x = \pm\pi/2$  creates a problem for any numerical integration technique which does not specifically account for the locations of these two singularities.

In this appendix, we will derive an analytic approximation to this truncated cosine, where the degree of fit can be as tight as desired. Actually, since in practice, the truncated  $\cos(x)$  response itself is not exact, an approximation to it is quite satisfactory and acceptable. It will immediately be obvious how the method extends to a truncated version of  $\cos^v(x)$ , where power  $v$  is arbitrary.

We begin by observing that

$$\cos(x) = \exp[\ln \cos(x)] \quad \text{for } |x| < \pi/2. \quad (\text{A-1})$$

Alternatively,

$$\cos(x) = \exp[\ln(1 - \{1 - \cos(x)\})] = \exp[\ln(1 - u)], \quad (\text{A-2})$$

where

$$u \equiv 1 - \cos(x). \quad (\text{A-3})$$

But, from the expansion

$$\ln(1 - u) = -\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \dots\right) \quad \text{for } |u| < 1, \quad (\text{A-4})$$

that is,  $|x| < \pi/2$ , we immediately obtain



$$\cos(x) = \exp\left[-\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \dots\right)\right] \quad \text{for } |x| < \frac{\pi}{2}, \quad (\text{A-5})$$

where  $u = 1 - \cos(x)$  from (A-3).

Now, suppose we terminate the infinite series in (A-5) at  $I$  terms, and use the approximation (where  $T\{\}$  denotes truncated)

$$T\{\cos(x)\} \equiv \exp\left[-\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \dots + \frac{1}{I}u^I\right)\right] \quad (\text{A-6})$$

for all  $x$  in the extended range  $[-\pi, \pi]$ , with  $u = 1 - \cos(x)$ . For large  $I$ , the right-hand side of (A-6) is a good approximation to  $\cos(x)$  for  $|x| < \pi/2$ . And, for  $|x| \geq \pi/2$ , where  $u \geq 1$ , the  $u$  series would tend to  $\infty$  as  $I \rightarrow \infty$ , meaning that the right-hand side of (A-6) is approximating zero, as desired. Furthermore, the right-hand side of (A-6) is always positive for all  $x$ , it has period  $2\pi$  in  $x$ , and it is smooth for all  $x$ . A sample plot of the left-hand side and right-hand side of (A-6) for  $I = 5$  is shown in figure A-1. The transition in behavior near  $x = \pm\pi/2$  is now smooth.

The approximation in (A-6) uses  $\cos(x)$  directly, namely, in the form  $u = 1 - \cos(x)$ . But,  $\cos(x)$  can be interpreted as the direction cosine between the element boresight angle and the angle of arrival. Thus, any other element response, that can be written directly in terms of this quantity, can thereby use (A-6).

For example, for a power of  $\cos(x)$ , there follows from (A-6),

$$T\{\cos^v(x)\} \equiv \exp\left[-v\left(u + \frac{1}{2}u^2 + \frac{1}{3}u^3 + \dots + \frac{1}{I}u^I\right)\right] \quad (\text{A-7})$$

for  $|x| \in [-\pi, \pi]$ , with  $u = 1 - \cos(x)$ .

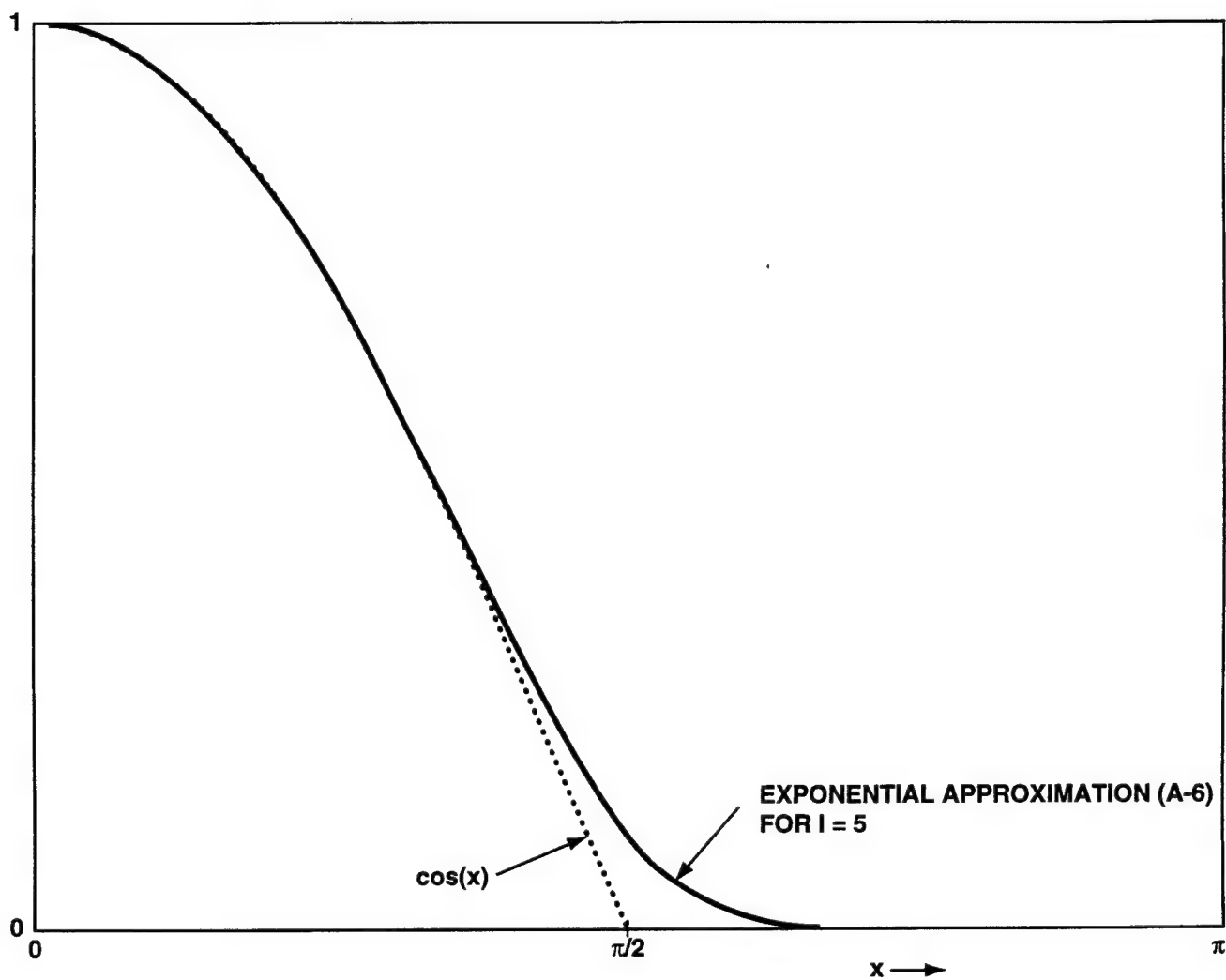


Figure A-1. Sample Plot of Equation A-6

## APPENDIX B — Program volumetric-grid.f

This program allows for arbitrary element locations  $\{x_\ell\}$  for  $1 \leq \ell \leq L$  and  $\{z_n\}$ ,  $\{y_n\}$  for  $1 \leq n \leq N$ . However, the particular example programmed utilizes equal spacing  $dx$  in the x-direction and equal arc lengths  $ds$  in the polar angle  $\phi$ , for simplicity. Both of these assumptions can easily be generalized to fit the users cases. Then, the two lines assigning values to  $dx$  and  $ds$  would be eliminated.

Coding for Hamming array weighting has not been included in this and the following appendix, since these array programs can utilize unequal spacings in the x- and/or z-directions. It is certainly possible to include nonuniform weighting, provided that the desired shading function is properly projected onto the actual unequal element locations, accounting for their density in space.

### COMMON INPUT PARAMETERS

$K = 400$ , number of azimuthal  $\theta$  samples over  $(0, 2\pi)$

$M = 192$ , number of polar  $\phi$  samples over  $(0, \pi)$

$f = 4,000$  Hz

$c = 4,900$  ft/sec

$\theta_s = \pi/2$ ,  $\phi_s = \pi/2$ ; broadside steering

$\Delta_x = 3.5$  inches,  $\Delta_z = 3.6$  inches for planar array

$\Delta_x = 3.5$  inches,  $\Delta_s = 3.6$  inches for volumetric array

### PROGRAM SYMBOL

### MATH SYMBOL AND EXPLANATION

lc	$L$ , number of elements in x-coordinate
nc	$N$ , number of elements in z-coordinate
kc	$K$ , number of $\theta$ samples over $(0, 2\pi)$
mc	$M$ , number of $\phi$ samples over $(0, \pi)$
ng	total number of gauss locations and weights
wx	storage of $L$ array weights for x-coordinate
wz	storage of $N$ array weights for z-coordinate
phis	$\phi_s$ , polar steering angle (radians)
thetas	$\theta_s$ , azimuthal steering angle (radians)
dphi	$\Delta\phi$ , phi increment in search for maximum
dtheta	$\Delta\theta$ , theta increment in search for maximum
j	gauss case number, $1 \leq j \leq 12$

```

implicit real*8(a-h,o-z)          ! volumetric-grid.f
real sec,dtime,time(1:2)
parameter(lc=200,nc=50,kc=400,ng=2520) ! kc = number of theta samples
dimension x(1:lc),y(1:nc),z(1:nc),psi(1:nc)
dimension wx(1:lc),wyz(1:nc),ax(1:lc),ayz(1:nc)
dimension xx(1:ng),ww(1:ng),c(1:kc),s(1:kc),cn(1:nc),sn(1:nc)
dimension mm(0:12)/0,16,24,32,48,64,96,128,192,256,384,512,768/
1  format(d28.18) ! case 1 2 3 4 5 6 7 8 9 10 11 12 <-- j
2  format(2d25.16)
3  format("sec",4e15.5)
pi=4.d0*atan(1.d0)

sec=dtime(time)                    ! use < legendre-array-dp
read 1, xx(1:ng)                   ! gauss-legendre locations
read 1, ww(1:ng)                   ! gauss-legendre weights
sec=dtime(time)
print 3, sec,time
print 2

ampd=.9d0                          ! relative power dip, .le. 1
phid=pi*.5d0                       ! polar angle of dip, radians
sigd=.1d0                          ! width of dip, radians

freq=4000.d0                       ! frequency, hertz
speed=4900.d0*12.d0                ! sound speed, inches/second
dx=3.5d0                           ! x spacing, inches
ds=3.6d0                           ! arc length spacing, inches
rc=200.d0                          ! radius of array, inches
a=0.d0                             ! lower limit on phi, radians
b=pi                               ! upper limit on phi, radians
phis=pi*.5d0                      ! polar steering angle, radians
thetas=pi*.5d0                    ! azimuth steering angle, radians
dphi=.0001d0                      ! phi increment in search, radians
dtheta=.0001d0                    ! theta increment in search, radians
j1=8                               ! starting gauss case, j1 .ge. 1
j2=10                             ! ending gauss case, j2 .le. 12

wavelength=speed/freq
cx=2.d0*pi/wavelength*dx
cyz=2.d0*pi/wavelength*rc

w1=0.d0
do n=1,lc
  x(n)=cx*n                        ! x element locations (arbitrary)
  wx(n)=1.d0                      ! x element weights (arbitrary)
  w1=w1+wx(n)
end do

w2=0.d0
t=(nc+1)*.5d0
da=ds/rc                          ! element angular spacing, radians
do n=1,nc
  an=da*(t-n)
  y(n)=cyz*cos(an)                ! y element locations (arbitrary)
  z(n)=cyz*sin(an)                ! z element locations (arbitrary)
  psi(n)=pi*.5d0-an              ! element polar angles in y,z plane
  wyz(n)=1.d0                    ! yz element weights (arbitrary)
  w2=w2+wyz(n)
end do
w3=w1*w1*w2*w2

t=sin(phis)
u=cos(thetas)*t
do n=1,lc
  ax(n)=x(n)*u
end do

```

```

t=sin(thetas)*t
u=cos(phis)
do n=1,nc
ayz(n)=y(n)*t+z(n)*u
end do

dt=2.d0*pi/kc          ! theta sampling increment
do k=1,kc
t=k*dt
c(k)=cos(t)
s(k)=sin(t)
end do

do n=1,nc
t=psi(n)
cn(n)=cos(t)
sn(n)=sin(t)
end do

ba1=(b+a)*.5d0
ba2=(b-a)*.5d0
m22=0
big=0.d0
t3=1.d0/3.d0

do j=1,12
mc=mm(j)                ! mc = number of phi samples
m22=m22+mm(j-1)
if (j .lt. j1) go to 9
if (j .gt. j2) go to 9
v=0.d0
two=0.d0

do m=1,mc
wm=ww(m+m22)
phim=ba1+ba2*xx(m+m22)
sm=sin(phim)
cm=cos(phim)
dm=(phim-phid)/sigd
dm=1.d0-ampd*exp(-.5d0*dm*dm)  ! noise power distribution
vk=0.d0
two=two+wm*sm*dm

do k=1,kc
cs=c(k)*sm
ss=s(k)*sm
ar=0.d0
ai=0.d0
br=0.d0
bi=0.d0

do n=1,lc
wn=wx(n)
t=x(n)*cs-ax(n)
ar=ar+wn*cos(t)
ai=ai+wn*sin(t)
end do !n

do n=1,nc
u=1.d0-sn(n)*ss-cn(n)*cm      ! element response:
rn=exp(-u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
wn=wyz(n)*rn
t=y(n)*ss+z(n)*cm-ayz(n)
br=br+wn*cos(t)
bi=bi+wn*sin(t)
end do !n

```

```

ab=(ar*ar+ai*ai)*(br*br+bi*bi)
vk=vk+ab
if (ab .lt. big) go to 10
big=ab
phib=phim
thetab=k
10 end do !k

v=v+wm*sm*dm*vk
end do !m

v=v*ba2*dt                                ! trapezoidal and gauss rules
two=two*ba2
dfw=two*2.d0*pi*w3/v
diw=10.d0*log10(dfw)
print 21, v
21 format("double integral",d25.16)
print 4, dfw,diw
4 format("dfw, diw",2d25.16)
dfb=two*2.d0*pi*big/v                      ! directivity factor
dib=10.d0*log10(dfb)                      ! directivity index
print 5, dfb,dib
5 format("dfb, dib",2d25.16)

sec=dtime(time)
print 3, sec,time
9 end do !j

thetab=thetab*dt
print 2
print 6, big,w3                            ! big = maximum power response
6 format("big, w3",2d25.16)
print 7, phib,thetab                      ! angles after coarse search
7 format("phib, thetab",2d25.16)

12 phio=phib                              ! fine search for maximum,
thetao=thetab                             ! starting from phib,thetab
do i=-1,1
phi=phio+dphi*i
cm=cos(phi)
sm=sin(phi)
do j=-1,1
theta=thetao+dtheta*j
cs=cos(theta)*sm
ss=sin(theta)*sm
ar=0.d0
ai=0.d0
br=0.d0
bi=0.d0

do n=1,lc
wn=wx(n)
t=x(n)*cs-ax(n)
ar=ar+wn*cos(t)
ai=ai+wn*sin(t)
end do !n

do n=1,nc
u=1.d0-sn(n)*ss-cn(n)*cm                ! element response:
rn=exp(-u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
wn=wyz(n)*rn
t=y(n)*ss+z(n)*cm-ayz(n)
br=br+wn*cos(t)
bi=bi+wn*sin(t)
end do !n

```

```

ab=(ar*ar+ai*ai)*(br*br+bi*bi)
if (ab .le. big) go to 11
big=ab
phib=phi
thetab=theta
11 end do !j
end do !i

if (abs(phib-phia)+abs(thetab-thetao) .gt. 0.d0) go to 12
print 2
print 8, big
8 format("big",2d25.16)
print 7, phib,thetab      ! angles after fine search
dfb=two*2.d0*pi*big/v      ! directivity factor
dib=10.d0*log10(dfb)      ! directivity index
print 5, dfb,dib

phib=phis      ! fine search for maximum,
thetab=thetas  ! starting from phis,thetas
big=0.d0
k=0
14 phio=phib
thetao=thetab
do i=-1,1
phi=phio+dphi*i
if (k .eq. 0) phi=phis
cm=cos(phi)
sm=sin(phi)
do j=-1,1
theta=thetao+dtheta*j
if (k .eq. 0) theta=thetas
cs=cos(theta)*sm
ss=sin(theta)*sm
ar=0.d0
ai=0.d0
br=0.d0
bi=0.d0

do n=1,lc
wn=wx(n)
t=x(n)*cs-ax(n)
ar=ar+wn*cos(t)
ai=ai+wn*sin(t)
end do !n

do n=1,nc
u=1.d0-sn(n)*ss-cn(n)*cm      ! element response:
rn=exp(-u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
wn=wyz(n)*rn
t=y(n)*ss+z(n)*cm-ayz(n)
br=br+wn*cos(t)
bi=bi+wn*sin(t)
end do !n

ab=(ar*ar+ai*ai)*(br*br+bi*bi)
if (ab .le. big) go to 13
big=ab
phib=phi
thetab=theta
if (k .eq. 0) go to 16
13 end do !j
end do !i
16 k=k+1
if (k .eq. 1) go to 14

```

```

15      if (abs(phib-phio)+abs(thetab-thetao) .gt. 0.d0) go to 14
        print 2
        print 8, big
        print 7, phib,thetab          ! angles near steering direction
        dfs=two*2.d0*pi*big/v         ! directivity factor
        dis=10.d0*log10(dfs)          ! directivity index
        print 15, dfs,dis
        format("dfs, dis",2d25.16)

        end

```



## APPENDIX C — Program planar-xz-grid.f

This program allows for arbitrary element locations  $\{x_\ell\}$  for  $1 \leq \ell \leq L$  and  $\{z_n\}$  for  $1 \leq n \leq N$ . However, the particular example programmed utilizes two equal spacings  $dx$  and  $dz$ , for simplicity; these assumptions can easily be generalized to fit the users cases. Then, the two input lines assigning values to  $dx$  and  $dz$  would be eliminated.

```

implicit real*8(a-h,o-z)          ! planar-xz-grid.f
real sec,dtime,time(1:2)
parameter(lc=200,nc=50,kc=400,ng=2520) ! kc = number of theta samples
dimension x(1:lc),z(1:nc),wx(1:lc),wz(1:nc),ax(1:lc),az(1:nc)
dimension xx(1:ng),ww(1:ng),c(1:kc),s(1:kc)
dimension mm(0:12)/0,16,24,32,48,64,96,128,192,256,384,512,768/
1 format(d28.18) ! case 1 2 3 4 5 6 7 8 9 10 11 12 <-- j
2 format(2d25.16)
3 format("sec",4e15.5)
pi=4.d0*atan(1.d0)

sec=dtime(time)                    ! use < legendre-array-dp
read 1, xx(1:ng)                   ! gauss-legendre locations
read 1, ww(1:ng)                   ! gauss-legendre weights
sec=dtime(time)
print 3, sec,time
print 2

ampd=.9d0                          ! relative power dip, .le. 1
phid=pi*.5d0                       ! polar angle of dip, radians
sigd=.1d0                          ! width of dip, radians

freq=4000.d0                       ! frequency, hertz
speed=4900.d0*12.d0               ! sound speed, inches/second
dx=3.5d0                          ! x spacing, inches
dz=3.6d0                          ! z spacing, inches
a=0.d0                            ! lower limit on phi, radians
b=pi                              ! upper limit on phi, radians
phis=pi*.5d0                      ! polar steering angle, radians
thetas=pi*.5d0                   ! azimuth steering angle, radians
dphi=.0001d0                      ! phi increment in search, radians
dtheta=.0001d0                   ! theta increment in search, radians
j1=8                              ! starting gauss case, j1 .ge. 1
j2=10                             ! ending gauss case, j2 .le. 12

wavelength=speed/freq
cx=2.d0*pi/wavelength*dx
cz=2.d0*pi/wavelength*dz

w1=0.d0
do n=1,lc
x(n)=cx*n                         ! x element locations (arbitrary)
wx(n)=1.d0                       ! x element weights (arbitrary)
w1=w1+wx(n)
end do

w2=0.d0
do n=1,nc
z(n)=cz*n                         ! z element locations (arbitrary)
wz(n)=1.d0                       ! z element weights (arbitrary)
w2=w2+wz(n)
end do
w3=w1*w1*w2*w2

u=cos(thetas)*sin(phis)
do n=1,lc
ax(n)=x(n)*u
end do
u=cos(phis)
do n=1,nc
az(n)=z(n)*u
end do

dt=2.d0*pi/kc                     ! theta sampling increment
do k=1,kc
t=k*dt                            ! theta(k)

```

```

c(k)=cos(t)
s(k)=sin(t)
end do

ba1=(b+a)*.5d0
ba2=(b-a)*.5d0
m22=0
big=0.d0
t3=1.d0/3.d0

do j=1,12
mc=mm(j)                ! mc = number of phi samples
m22=m22+mm(j-1)
if (j .lt. j1) go to 9
if (j .gt. j2) go to 9
v=0.d0
two=0.d0

do m=1,mc
wm=ww(m+m22)
phim=ba1+ba2*xx(m+m22)
sm=sin(phim)
cm=cos(phim)
dm=(phim-phid)/sigd
dm=1.d0-ampd*exp(-.5d0*dm*dm)    ! noise power distribution
vk=0.d0
two=two+wm*sm*dm
br=0.d0
bi=0.d0

do n=1,nc
wn=wz(n)
t=z(n)*cm-az(n)
br=br+wn*cos(t)
bi=bi+wn*sin(t)
end do !n
bsq=br*br+bi*bi

do k=1,kc
cs=c(k)*sm
ss=s(k)*sm
u=1.d0-ss                ! element response:
rsq=exp(-2.d0*u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
ar=0.d0
ai=0.d0

do n=1,lc
wn=wx(n)
t=x(n)*cs-ax(n)
ar=ar+wn*cos(t)
ai=ai+wn*sin(t)
end do !n

ab=rsq*(ar*ar+ai*ai)*bsq
vk=vk+ab
if (ab .lt. big) go to 10
big=ab
phib=phim
thetab=k
end do !k

v=v+wm*sm*dm*vk
end do !m

v=v*ba2*dt                ! trapezoidal and gauss rules
two=two*ba2

```

```

dfw=two*2.d0*pi*w3/v
diw=10.d0*log10(dfw)
print 21, v
21 format("double integral",d25.16)
print 4, dfw,diw
4 format("dfw, diw",2d25.16)
dfb=two*2.d0*pi*big/v          ! directivity factor
dib=10.d0*log10(dfb)          ! directivity index
print 5, dfb,dib
5 format("dfb, dib",2d25.16)

sec=dttime(time)
print 3, sec,time
9 end do !j

thetab=thetab*dt
print 2
print 6, big,w3                ! big = maximum power response
6 format("big, w3",2d25.16)
print 7, phib,thetab          ! angles after coarse search
7 format("phib, thetab",2d25.16)

12 phio=phib                    ! fine search for maximum,
thetao=thetab                  ! starting from phib,thetab
do i=-1,1
phi=phio+dphi*i
cm=cos(phi)
sm=sin(phi)
do j=-1,1
theta=thetao+dtheta*j
cs=cos(theta)*sm
ss=sin(theta)*sm
u=1.d0-ss                      ! element response:
rsq=exp(-2.d0*u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
ar=0.d0
ai=0.d0
br=0.d0
bi=0.d0

do n=1,lc
wn=wx(n)
t=x(n)*cs-ax(n)
ar=ar+wn*cos(t)
ai=ai+wn*sin(t)
end do !n

do n=1,nc
wn=wz(n)
t=z(n)*cm-az(n)
br=br+wn*cos(t)
bi=bi+wn*sin(t)
end do !n

ab=rsq*(ar*ar+ai*ai)*(br*br+bi*bi)
if (ab .le. big) go to 11
big=ab
phib=phi
thetab=theta
11 end do !j
end do !i

if (abs(phib-phio)+abs(thetab-thetao) .gt. 0.d0) go to 12
print 2
print 8, big
8 format("big",2d25.16)
print 7, phib,thetab          ! angles after fine search

```

```

dfb=two*2.d0*pi*big/v      ! directivity factor
dib=10.d0*log10(dfb)       ! directivity index
print 5, dfb,dib

phib=phis                  ! fine search for maximum,
thetab=thetas              ! starting from phis,thetas
big=0.d0
k=0
14 phio=phib
thetao=thetab
do i=-1,1
phi=phio+dphi*i
if (k .eq. 0) phi=phis
cm=cos(phi)
sm=sin(phi)
do j=-1,1
theta=thetao+dtheta*j
if (k .eq. 0) theta=thetas
cs=cos(theta)*sm
ss=sin(theta)*sm
u=1.d0-ss                  ! element response:
rsq=exp(-2.d0*u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
ar=0.d0
ai=0.d0
br=0.d0
bi=0.d0

do n=1,lc
wn=wx(n)
t=x(n)*cs-ax(n)
ar=ar+wn*cos(t)
ai=ai+wn*sin(t)
end do !n

do n=1,nc
wn=wz(n)
t=z(n)*cm-az(n)
br=br+wn*cos(t)
bi=bi+wn*sin(t)
end do !n

ab=rsq*(ar*ar+ai*ai)*(br*br+bi*bi)
if (ab .le. big) go to 13
big=ab
phib=phi
thetab=theta
if (k .eq. 0) go to 16
end do !j
end do !i
16 k=k+1
if (k .eq. 1) go to 14

if (abs(phib-phio)+abs(thetab-thetao) .gt. 0.d0) go to 14
print 2
print 8, big
print 7, phib,thetab      ! angles near steering direction
dfs=two*2.d0*pi*big/v     ! directivity factor
dis=10.d0*log10(dfs)      ! directivity index
print 15, dfs,dis
15 format("dfs, dis",2d25.16)

end

```

## APPENDIX D — Program planar-xz-equal.f

In this appendix, coding for Hamming array weighting has been included in the listed program (as an option) for either or both of the x- and z-coordinates. However, it has been exercised in only two of the numerical examples that we present here.

```

implicit real*8(a-h,o-z)      ! planar-xz-equal.f
real sec,dtime,time(1:2)
parameter(lc=200,nc=50,kc=400,ng=2520) ! kc = number of theta samples
dimension xx(1:ng),ww(1:ng),wx(1:lc),wz(1:nc),c(1:kc),s(1:kc)
dimension mm(0:12)/0,16,24,32,48,64,96,128,192,256,384,512,768/
1 format(d28.18) ! case 1 2 3 4 5 6 7 8 9 10 11 12 <-- j
2 format(2d25.16)
3 format("sec",4e15.5)
pi=4.d0*atan(1.d0)

sec=dtime(time)      ! use < legendre-array-dp
read 1, xx(1:ng)      ! gauss-legendre locations
read 1, ww(1:ng)      ! gauss-legendre weights
sec=dtime(time)
print 3, sec,time
print 2

ampd=.9d0      ! relative power dip, .le. 1
phid=pi*.5d0   ! polar angle of dip, radians
sigd=.1d0      ! width of dip, radians

freq=4000.d0   ! frequency, hertz
speed=4900.d0*12.d0 ! sound speed, inches/second
dx=3.5d0       ! x spacing, inches
dz=3.6d0       ! z spacing, inches
a=0.d0         ! lower limit on phi, radians
b=pi           ! upper limit on phi, radians
phis=pi*.5d0   ! polar steering angle, radians
thetas=pi*.5d0 ! azimuth steering angle, radians
dphi=.0001d0   ! phi increment in search, radians
dtheta=.0001d0 ! theta increment in search, radians
j1=8           ! starting gauss case, j1 .ge. 1
j2=10          ! ending gauss case, j2 .le. 12

wavelength=speed/freq
cx=2.d0*pi/wavelength*dx
cz=2.d0*pi/wavelength*dz
bx=cx*cos(thetas)*sin(phis)
bz=cz*cos(phis)

w1=0.d0
t=(lc+1)*.5d0
if (lc .gt. 1) go to 22
u=0.d0
go to 23
22 u=2.d0*pi/(lc-1)
23 do n=1,lc
c wx(n)=.54d0+.46d0*cos(u*(n-t)) ! x element weights (Hamming)
wx(n)=1.d0 ! x element weights (arbitrary)
w1=w1+wx(n)
end do

w2=0.d0
t=(nc+1)*.5d0
if (nc .gt. 1) go to 24
u=0.d0
go to 25
24 u=2.d0*pi/(nc-1)
25 do n=1,nc
c wz(n)=.54d0+.46d0*cos(u*(n-t)) ! z element weights (Hamming)
wz(n)=1.d0 ! z element weights (arbitrary)
w2=w2+wz(n)
end do
w3=w1*w1*w2*w2

dt=2.d0*pi/kc      ! theta sampling increment

```

```

do k=1,kc
t=k*dt
c(k)=cos(t)
s(k)=sin(t)
end do

ba1=(b+a)*.5d0
ba2=(b-a)*.5d0
m22=0
big=0.d0
t3=1.d0/3.d0

do j=1,12
mc=mm(j)
m22=m22+mm(j-1)
if (j .lt. j1) go to 9
if (j .gt. j2) go to 9
v=0.d0
two=0.d0

do m=1,mc
wm=ww(m+m22)
phim=ba1+ba2*xx(m+m22)
sm=sin(phim)
fx=cx*sm
ez=cz*cos(phim)-bz
dm=(phim-phid)/sigd
dm=1.d0-ampd*exp(-.5d0*dm*dm) ! noise power distribution
vk=0.d0
two=two+wm*sm*dm
br=0.d0
bi=0.d0

r=1.d0
q=0.d0
ce=cos(ez)
se=sin(ez)
do n=1,nc
t=r*ce-q*se
q=q*ce+r*se
r=t
wn=wz(n)
br=br+wn*r
bi=bi+wn*q
end do !n
bsq=br*br+bi*bi

do k=1,kc
u=1.d0-s(k)*sm
rsq=exp(-2.d0*u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
ex=fx*c(k)-bx
ar=0.d0
ai=0.d0

r=1.d0
q=0.d0
ce=cos(ex)
se=sin(ex)
do n=1,lc
t=r*ce-q*se
q=q*ce+r*se
r=t
wn=wx(n)
ar=ar+wn*r
ai=ai+wn*q
end do !n

```



```

ab=rsq*(ar*ar+ai*ai)*bsq
vk=vk+ab
if (ab .lt. big) go to 10
big=ab
phib=phim
thetab=k
10 end do !k

v=v+wm*sm*dm*vk
end do !m

v=v*ba2*dt                                ! trapezoidal and gauss rules
two=two*ba2
dfw=two*2.d0*pi*w3/v
diw=10.d0*log10(dfw)
print 21, v
21 format("double integral",d25.16)
print 4, dfw,diw
4 format("dfw, diw",2d25.16)
dfb=two*2.d0*pi*big/v                      ! directivity factor
dib=10.d0*log10(dfb)                       ! directivity index
print 5, dfb,dib
5 format("dfb, dib",2d25.16)

sec=dtime(time)
print 3, sec,time
9 end do !j

thetab=thetab*dt
print 2
print 6, big,w3                            ! big = maximum power response
6 format("big, w3",2d25.16)
print 7, phib,thetab                      ! angles after coarse search
7 format("phib, thetab",2d25.16)

12 phio=phib                               ! fine search for maximum,
thetao=thetab                             ! starting from phib,thetab
do i=-1,1
phi=phio+dphi*i
sm=sin(phi)
fx=cx*sm
ez=cz*cos(phi)-bz
br=0.d0
bi=0.d0

r=1.d0
q=0.d0
ce=cos(ez)
se=sin(ez)
do n=1,nc
t=r*ce-q*se
q=q*ce+r*se
r=t
wn=wz(n)
br=br+wn*r
bi=bi+wn*q
end do !n
bsq=br*br+bi*bi

do j=-1,1
theta=thetao+dtheta*j
u=1.d0-sin(theta)*sm                      ! element response:
rsq=exp(-2.d0*u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))
ex=fx*cos(theta)-bx
ar=0.d0

```

```

ai=0.d0

r=1.d0
q=0.d0
ce=cos(ex)
se=sin(ex)
do n=1,lc
t=r*ce-q*se
q=q*ce+r*se
r=t
wn=wx(n)
ar=ar+wn*r
ai=ai+wn*q
end do !n

ab=rsq*(ar*ar+ai*ai)*bsq
if (ab .le. big) go to 11
big=ab
phib=phi
thetab=theta
11 end do !j
end do !i

if (abs(phib-phio)+abs(thetab-thetao) .gt. 0.d0) go to 12
print 2
print 8, big
8 format("big",2d25.16)
print 7, phib,thetab          ! angles after fine search
dfb=two*2.d0*pi*big/v         ! directivity factor
dib=10.d0*log10(dfb)          ! directivity index
print 5, dfb,dib

phib=phis                      ! fine search for maximum,
thetab=thetas                  ! starting from phis,thetas
big=0.d0
k=0
14 phio=phib
thetao=thetab
do i=-1,1
phi=phio+dphi*i
if (k .eq. 0) phi=phis
sm=sin(phi)
fx=cx*sm
ez=cz*cos(phi)-bz
br=0.d0
bi=0.d0

r=1.d0
q=0.d0
ce=cos(ez)
se=sin(ez)
do n=1,nc
t=r*ce-q*se
q=q*ce+r*se
r=t
wn=wz(n)
br=br+wn*r
bi=bi+wn*q
end do !n
bsq=br*br+bi*bi

do j=-1,1
theta=thetao+dtheta*j
if (k .eq. 0) theta=thetas
u=1.d0-sin(theta)*sm          ! element response:
rsq=exp(-2.d0*u*(1.d0+u*(.5d0+u*(t3+u*(.25d0+u*.2d0))))

```

```

ex=fx*cos(theta)-bx
ar=0.d0
ai=0.d0

r=1.d0
q=0.d0
ce=cos(ex)
se=sin(ex)
do n=1,lc
t=r*ce-q*se
q=q*ce+r*se
r=t
wn=wx(n)
ar=ar+wn*r
ai=ai+wn*q
end do !n

ab=rsq*(ar*ar+ai*ai)*bsq
if (ab .le. big) go to 13
big=ab
phib=phi
thetab=theta
if (k .eq. 0) go to 16
13 end do !j
end do !i
16 k=k+1
if (k .eq. 1) go to 14

if (abs(phib-phia)+abs(thetab-thetao) .gt. 0.d0) go to 14
print 2
print 8, big
print 7, phib,thetab          ! angles near steering direction
dfs=two*2.d0*pi*big/v         ! directivity factor
dis=10.d0*log10(dfs)          ! directivity index
print 15, dfs,dis
15 format("dfs, dis",2d25.16)

end

```

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